

On Ilyashenko's Statistical Attractors

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Abstract

We revisit the notion of statistical attractor defined by Ilyashenko. We argue to show that they are optimally defined to exist and to describe the asymptotical statistics of Lebesgue-almost all the orbits. We contribute to the theory, defining a minimality concept of α -observable statistical attractors and proving that the space is always full Lebesgue decomposable into pairwise disjoint sets that are Lebesgue-bounded away from zero and included in the basins of a finite family of minimally observable statistical attractors. We illustrate the abstract theory including, among other examples, the Bowen homeomorphisms with non robust topological heteroclinic cycles. We prove the existence of three types of statistical behaviors for these examples.

Keywords: Statistical attractors; SRB measures; physical measures.

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1 Overview

In [1] [2], Milnor introduced a definition of attractor, describing the asymptotic topological behavior of the Lebesgue-probable orbits. Precisely, the approximation to a Milnor's attractor is achieved for all the instants large enough, while the probabilistic ingredient appears only in the choice of the initial state x , which is distributed on the ambient manifold M according to the Lebesgue measure (see Definition 6.1). As Milnor remarks in [1], his definition of attractor is broad enough to ensure the existence of at least one attractor for any continuous system f on the compact manifold M (Lemma 2 of [1]). Sometimes in papers, the words “global Milnor's attractor” mean what Milnor calls the “likely limit set”: the least closed set in the manifold containing the omega-limit set of the trajectories for Lebesgue-almost all the initial

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states. It is minimal in the sense that it is a global attractor which is the smallest in the chain of inclusion.

A non global meaning of a Milnor's attractor is also standard: it is a closed set that contains the omega-limit set of a Lebesgue-positive set of orbits. Nevertheless, in its non global sense, a minimal Milnor's attractor may not exist. It could not exist a topological non-global attractor such that none of its proper subsets is also a topological attractor for a Lebesgue-positive set of orbits. There may exist a chain of smaller and smaller sets, attracting smaller and smaller parts of the phase space, all with positive Lebesgue measure, but such that their intersection attracts measure zero. This is the problem due to which a system can have no minimal attractors (in the sense of a least set that attracts something with positive Lebesgue measure). In spite of that, we solve easily this problem: one can apply Lemma 1 of [1] to prove that for any fixed $0 < \alpha \leq 1$ there exists a minimal α -observable Milnor's attractor: its basin of topological attraction has Lebesgue measure larger or equal than $\alpha > 0$ (see Definition 6.1).

From a different viewpoint, and with a different mathematical purpose, in [3] Pugh and Shub call ergodic attractor to the support K of an ergodic hyperbolic measure μ that has certain property of absolute continuity respect to the internal Lebesgue measures of the leaves of an unstable foliation. Such a set K is called ergodic attractor because it is the support of the measure μ which is ergodic and, in addition, physical or Sinai-Ruelle-Bowen (SRB) (see [4], [5] and [6]). Namely, μ is the weak*-limit, for a Lebesgue-positive set of initial states $x \in M$, of the empirical probabilities $\nu_n(x)$ supported on finite pieces of the future orbit of x . Precisely: $\mu = \lim_{n \rightarrow +\infty}^* \nu_n(x)$, or equivalently $\mathcal{L}^*(x) = \{\mu\}$, where $\nu_n(x)$ and $\mathcal{L}^*(x)$ are defined as follows:

Definition.

The *sequence of empirical probabilities* of the orbit with initial state $x \in M$ is

$$\{\nu_n(x)\}_{n \geq 1}, \quad \text{where} \quad \nu_n(x) := \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \in \mathcal{M}, \quad (1)$$

being δ_y the Dirac delta supported on the point $y \in M$. In other words, $\nu_n(x)$ is the probability measure supported, and equally distributed, on the finite piece of the orbit of x between the instant 0 and the instant $n - 1$.

The *limit set in the space of probabilities* \mathcal{M} of the future orbit with initial state in the point $x \in M$ is:

$$\mathcal{L}^*(x) := \{\mu \in \mathcal{M} : \exists n_i \rightarrow +\infty \text{ such that } \lim_{i \rightarrow +\infty}^* \nu_{n_i}(x) = \mu\}. \quad (2)$$

The symbol \lim^* denotes the limit in \mathcal{M} endowed with the weak* topology. Since \mathcal{M} is compact and sequentially compact, then $\mathcal{L}^*(x)$ is nonempty, and weak* closed

and compact, for all $x \in M$. We say that $\mathcal{L}^*(x)$ describes *the asymptotic statistics* of the future orbit with initial state x .

In general we call a (non necessarily ergodic) measure μ physical or SRB if $\mathcal{L}^*(x) = \{\mu\}$ for a Lebesgue-positive set $B(\mu)$ of initial states $x \in M$ (see also Definition 6.2 in Section 7). Therefore, a physical or SRB measure μ is a probabilistic spacial description of the asymptotic statistics of the orbits in $B(\mu)$. The set $B(\mu) \subset M$ is called basin of statistical attraction of μ . By definition, we agree to say that a compact f -invariant set K in the ambient manifold M is an ergodic attractor if it is the minimal compact support of a physical SRB measure μ , even if μ is non ergodic (see Definition 6.2).

Notice that all the continuous systems on a compact manifold M do exhibit ergodic invariant measures, and that the basin of statistical attraction of each ergodic measure μ is nonempty. In fact, it includes μ almost every point, after the definition of ergodicity of μ . But it can happen that none of the invariant measures is physical or SRB, because their basins of statistical attraction have all zero Lebesgue measure. So, the continuity of f does not guarantee the existence of ergodic attractors. The study of the asymptotic statistics of Lebesgue-almost all the orbits is not always possible if considering only the definition of physical or SRB measures.

We remark that physical SRB measures are particularly important, when they exist, for systems that do not preserve the Lebesgue measure m , or for which m is non ergodic. Their relevance reside in the probabilistic spatial description of the asymptotic statistics of the system restricted to their basins. The problem of existence of physical SRB measures is one of the non trivial differences between the ergodic and the Milnor's attractors. On the one hand, Milnor's attractors do always exist for any continuous map. On the other hand, even for C^∞ maps, physical measures may not exist, and thus no ergodic attractor may be exhibited. The description of those systems that have ergodic attractors (or physical SRB measures) is a major problem of the modern Differentiable Ergodic Theory, and is mostly developed for systems that are C^1 plus Hölder and have some kind of hyperbolic behavior ([7] [8] [9]).

The other non trivial difference between ergodic and Milnor's attractors is that the first ones are usually finer than the second ones. This fineness is detected if the experimenter aims to observe the asymptotic statistics of probable orbits, and not only the spacial form of their topological attractors or limit sets. Namely, assume that the purpose is to study probabilistically which regions of the space (say small neighborhoods in M) are visited asymptotically with positive frequency of future iterates from a m -positive set of initial states. Then, the knowledge of the topological strong approach to a compact Milnor's attractor K is not enough (we will see examples in the paragraphs 7.1, 7.2 and 7.3). In fact, the topological approach to K can not distinguish the frequency of visits of the orbits to the small neighborhoods of the different points of K . To have a view of this phenomenon,

assume that K is a Milnor's attractor whose basin of topological attraction is A . The set K may contain points y that are irrelevant for the statistical observation: for Lebesgue almost all the initial states $x \in A$ the frequency of visits to any small neighborhood of y may be asymptotically null, even if there exist visits for arbitrarily large times (and so y belongs to the omega-limit of x). In this sense, an ergodic attractor, defined as the compact support of a physical or SRB probability μ , is a finer concept than the Milnor's attractor: it excludes from the attractor those statistically irrelevant points. But it is excessively finer, and so some interesting continuous systems do not exhibit ergodic attractors (see Example 7.2, case (C)).

Inspired both in the Milnor and in the ergodic attractors, one would like to consider a kind of attractor that shares the advantages of both definitions: the sure existence of Milnor's attractors and the fine statistical description of the ergodic attractors. In fact, that kind of "good" attractors was defined and studied in the last 15 years: In [10], [11], [12], Ilyashenko and other authors had introduced the concept of "statistical attractors". The definition appears also in [13] [14] [15], and its relation with the time averages of functions is treated in [16].

We give here (with a slightly different look) a definition of statistical attractor which is equivalent to that of Ilyashenko and his co-authors. We add the concept of α -observability of the statistical attractors, that will be useful to obtain a minimality property and prove Theorem 2.3 and its corollaries, as our contribution to the theory.

In the sequel the Lebesgue measure m is normalized to be a probability measure on the manifold M .

Definition 1.1 (STATISTICAL ATTRACTOR)

Let M be a compact manifold. Let $f: M \rightarrow M$ be a Borel measurable map. Let $K \subset M$ be a nonempty, compact and f -invariant set, i.e. $f^{-1}(K) = K$. We say that K is a *statistical attractor* if the following set $B(K)$ (which we call *basin of statistical attraction* of K , or in brief, *basin* of K) has positive Lebesgue measure:

$$B(K) := \{x \in M : \lim_{n \rightarrow +\infty} \frac{1}{n} \#\{0 \leq j \leq n-1 : \text{dist}(f^j(x), K) < \epsilon\} = 1 \quad \forall \epsilon > 0\},$$

where $\#A$ denotes the cardinality of the finite set A .

For any $0 < \alpha \leq 1$, we say that the statistical attractor K is α -*observable* if $m(B(K)) \geq \alpha$, where m denotes the Lebesgue measure. We abbreviate this property by α -*obs*.

We say that an statistical attractor is *minimal* α -*obs*, if it is α -obs. and has no proper subsets that are also α -obs statistical attractors for the same value of α .

Let us explain the definition above. For any chosen initial state $x \in M$ and for any chosen natural number $n \geq 1$, the set $\{0 \leq j \leq n-1 : \text{dist}(f^j(x), K) < \epsilon\}$ is composed by all the return times j (between the instant 0 and up to the instant $n-1$) of the future orbit with initial state x to the ϵ -neighborhood of the attractor

K . Thus, the quotient $\#\{0 \leq j \leq n-1 : \text{dist}(f^j(x), K) < \epsilon\} / n$ is the frequency according to which the orbit of x , during the time interval $[0, n-1]$, visited the ϵ -neighborhood of K . Therefore, the orbit of a point $x \in M$ belongs to the basin of statistical attraction $B(K)$ of K , if and only if the asymptotic frequency of its visits in the future to any arbitrarily small neighborhood of K is 1. The future orbit does not need to stay near K in all the instants that are large enough. We only require that the frequency (i.e. the probability), according to which one may find a future iterate of x far from K , is asymptotically null. So, the attraction to K is not necessarily topological as in the Milnor's attractors, but statistical as in the Pugh-Shub ergodic attractors.

Finally, notice that K is a statistical attractor only if its basin $B(K)$ of statistical attraction has positive Lebesgue measure. Thus, a nonempty basin $B(K)$, as the compact support of any ergodic measure has, is not enough.

In brief, Definition 1.1 of statistical attractor combines the notions of Milnor's attractor and of ergodic attractor (see Definitions 6.1 and 6.2 in Section 7). This combination is not arbitrary, but designed in such a way that statistical attractors inherit jointly the previously separated advantages of both. In fact, on the one hand statistical attractors exist for all the continuous systems, as Milnor attractors do (see Theorem 2.1 in Section 2). On the other hand, statistical attractors describe finely the asymptotic statistics of a positive Lebesgue set of initial states, as ergodic attractors do when they exist (see Theorem 2.2 in Section 2).

It is immediate that any Milnor's attractor (according to [1], which we revisit in Definition 6.1) is also a statistical attractor satisfying Definition 1.1. But, as we show in Examples 7.1 and 7.2 of Section 7, not all the minimal α -obs Milnor attractors are minimal α -obs statistical. Nevertheless, as a corollary of Theorem 2.3 (see the end of Section 2) we prove the following statement: The basin $A(K)$ of topological attraction of any α -obs Milnor's attractor K , is the union (up to a zero Lebesgue measure set) of the basins $B(K_i)$ of statistical attraction of a finite family of minimal α_i -obs statistical attractors $K_i \subset K$, for some adequate positive values of α_i . In the above result, the union of all the minimal statistical attractors K_i contained in K , is not necessarily equal to the Milnor's attractor K (see Examples 7.1 and 7.2 in Section 7). Therefore, the statistical attractors are thinner sets than the Milnor's attractors.

To end this overview, we must notice that along this paper, we are using the adjective " α -obs. minimal" in the sense of a least set in the chain of inclusions that is an α -observable statistical attractor for a fixed value of α . Nevertheless, there is a concept of "minimal attractor" (that we do not treat along this paper), which was defined also by Ilyashenko [17]. It is also referring to a statistical attraction, but minimal attractors in [17] do not necessarily coincide with α -obs. minimal statistical attractors. On purpose, the Ph.D. thesis of A. Gorodetski was devoted to the study of different definitions of attractors and the relations between them (inclusion,

coincidence, coincidence for hyperbolic maps, and other properties). Some parts of this thesis are published in [11]. For the minimal attractors, see also Bachurin's article [16] and for a proof that they may not coincide with the statistical attractors, see Kleptsyn's article [14]. For both minimal and statistical attractors there is a known interplay among their direct definitions, their definitions via negligible sets [10]; their definitions via the supports of limit measures for the time average (Krylov-Bogolyubov procedure), and the time averages of functions [16]¹.

2 Statement of the results

In this section we state the main theorems that are proved along the paper. The results of theorems 2.1 and 2.2 are already known. The proof of Theorem 2.1 is standard. We include it for a seek of completeness, since it is adapted to our slightly different (but equivalent) formulation in Definition 1.1 of the statistical attractors, with respect to the standard definitions. The proof of Theorem 2.2 is rather different from the standard ones. It still uses the KrylovBogolyubov procedure to construct invariant measures that are statistically significative, and to deduce that the statistical attractors support those good measures. But, the arguments along the proofs are based in the definition and properties of SRB-like measures which were introduced in [18]. Finally, Theorem 2.3 which proves a natural decomposition of a Lebesgue-full set in the phase space into different basins of statistical attraction is contribution of this paper to the theory, jointly with the concept of minimal α -observable statistical attractors in the second part of Definition 1.1.

Theorem 2.1 (EXISTENCE OF STATISTICAL ATTRACTORS)

Let $f: M \rightarrow M$ be a Borel-measurable map on a compact manifold M of finite dimension. Then, for all $0 < \alpha \leq 1$ there exist minimal α -observable statistical attractors according to Definition 1.1.

Moreover, if $\alpha = 1$, then the minimal α -obs. statistical attractor is unique.

We prove this Theorem in Section 3. In that section we also state and prove Theorems 3.3 and 3.4, that are slight generalizations of Theorem 2.1 relative to some previously fixed invariant subsets of the phase space.

It is standard to check that if a physical SRB measure μ exists, then its compact support K is a statistical attractor. In this sense statistical attractors are the natural generalization of ergodic attractors (being these latter, when they exist, the f -invariant compact sets that satisfy Definition 6.2). The following Theorem 2.2 states a weak converse property. On the one hand we have that the minimal compact support of a physical SRB measure, when it exists, is an α -obs. minimal

¹Most of this paragraph was taken, with slight changes, from a report of an anonymous referee.

statistical attractor for some $\alpha > 0$. On the other hand Theorem 2.2 asserts that any α -obs minimal statistical attractor is the minimal compact support of a set of physical-like or SRB-like measures. Those measures are obtained after applying the Krylov-Bogolyubov procedure to the empirical probabilities constructed in Equality 1. Indeed, this procedure consists in taking any weak* partial limit of the time averages of non necessarily invariant probabilities.

Theorem 2.2 (CHARACTERIZATION OF STATISTICAL ATTRACTORS)

If K is an α -observable statistical attractor for some $0 < \alpha \leq 1$, and if $B(K)$ is its basin of attraction, then there exists a unique non empty weak-compact set $\mathcal{O}_f(K)$ of probability measures (which we call physical-like or SRB-like measures) such that:*

(a) *For Lebesgue almost all the initial states $x \in B(K)$, and for all the convergent subsequences of the empirical distributions*

$$\nu_n(x) := (1/n) \sum_{j=0}^{n-1} \delta_{f^j(x)},$$

their weak-limits are probabilities contained in $\mathcal{O}_f(K)$.*

(b) *$\mathcal{O}_f(K)$ is the minimal weak*-compact set of probabilities satisfying (a).*

(c) *If besides K is minimal containing its basin of attraction $B(K)$ or if K is α -obs minimal restricted to $B(K)$, in particular if K is α -obs minimal in the whole ambient manifold M , then K is the common compact support of all the probabilities in $\mathcal{O}_f(K)$, i.e. K is the minimal compact set in M such that $\mu(K) = 1 \forall \mu \in \mathcal{O}_f(K)$.*

We prove Theorem 2.2 in Section 4. In the proof we use the definition of SRB-like (or physical-like) measures defined in [18] and take the main steps of the argument from [18], [19]. For a seek of completeness, we include in this paper Definition 4.1 of the SRB-like measures. They are generalizations of the SRB measures, of the observable measures defined in [20], and of the statistical asymptotic measures of [21]. In the proof of Theorem 2.2, we revisit, reproduce or adapt some of the arguments in [18], [19], in particular those showing the existence and optimality of the SRB-like measures.

Theorem 2.2 implies that for all $0 < \alpha \leq 1$, any α -observable statistical attractor K is provided with a minimal weak*-compact subset $\mathcal{O}_f(K)$ of probability measures that has two remarkable properties:

(1) It is set of f -invariant measures which have, in relation with the attractor K , the same “physical” role as physical measures have, when they exist, in relation to the ergodic attractors. In fact, after the statement (a) of Theorem 2.2, the invariant measures in \mathcal{O}_f completely describe the asymptotic statistics of Lebesgue-almost all the orbits attracted by K .

(2) It is the optimal set of probability measures that completely describes the asymptotic statistics, as stated in part (b) of Theorem 2.2. Therefore, the statistical attractors are the optimal choice, among the compact invariant sets in the ambient manifold M , if one aims to describe completely the asymptotic statistics of Lebesgue almost all the orbits in their basins.

In the following Theorem 2.3, we state the existence of a decomposition of the space, up to a zero-Lebesgue subset, into a finite family of sets, each one contained in the basin of attraction of a statistical attractor satisfying a relative minimality condition. Thus, after Theorem 2.2, this result implies the existence of probabilistically spacial descriptions of the asymptotic dynamical statistics, for Lebesgue almost all initial states.

Theorem 2.3 (FINITE DECOMPOSITION INTO STATISTICAL ATTRACTORS)

Let $0 < \alpha \leq 1$ be fixed. Let m denote the Lebesgue probability measure.

There exists a finite family $\{K_i\}_{1 \leq i \leq p}$ of α_i -obs statistical attractors K_i with basins $B(K_i)$ such that:

(a) $\bigcup_{i=1}^p B(K_i)$ covers m -almost all M .

(b) $\alpha_i = \alpha$ for all the values of $i \in \{1, \dots, p\}$ except at most one, say i_0 , for which $0 < \alpha_{i_0} = m(B(K_{i_0})) < \alpha$.

(Therefore $m(B(K_i)) \geq \alpha \ \forall i \in \{1, \dots, p\}$ such that $i \neq i_0$.)

(c) For all $1 \leq i \leq p$ the statistical attractor K_i is α_i -obs. minimal restricted to $M \setminus \bigcup_{j=1}^{i-1} B(K_j)$ according to Definition 3.1. (We denote $\bigcup_{j=1}^0 \cdot = \emptyset$.)

We prove Theorem 2.3 in Section 5. To prove it, we introduce and use the concept of α -observability, to get minimality conditions to the statistical attractors, and then we apply the standard ideas of the proofs of Theorems 2.1 and 2.2. The statement and the proof of Theorem 2.3 is rather natural: roughly speaking, one can take away minimal observable sets (together with what they attract), one by one.

We notice that the statistical attractors K_i of the decomposition in Theorem 2.3 are not necessarily pairwise disjoint. So, also their basins of attraction $B(K_i)$ are not necessarily pairwise disjoint. If all the statistical attractors K_i were mutually disjoint, then any pair of them would be at positive distance (since they are compact sets), and so, it is immediate that their basins would be also pairwise disjoint. So, if this additional assumption holded, Theorem 2.3 would assert that the basins $B(K_i)$ of the finitely many statistical attractors K_i would form a partition of Lebesgue-a.e. the phase space. Anyway, even if the disjointness condition did not hold, the basins of attractions of the finite number of α -obs minimal statistical attractors cover Lebesgue-a.e. the space, and are, one by one, Lebesgue-bounded away from zero.

To end this section, we deduce an immediate corollary of Theorem 2.3, which shows that the statistical attractors are more sensible than Milnor's attractors:

namely, each α -obs. minimal Milnor's attractor contains the union of a finite number of statistical attractors

Corollary of Theorem 2.3

Let $0 < \alpha \leq 1$, and let K be an α -obs. minimal Milnor's attractor with basin $A(K)$ (according to Definition 6.1).

There exists a finite number of statistical attractors K_1, \dots, K_p contained in K that satisfy the conditions (a), (b) and (c) of Theorem 2.3 for the set $A(K)$ instead of M .

This corollary is immediate after Theorem 2.3. In fact, along the proof of Theorem 2.3 one does not need the manifold structure of the ambient space M for any purpose except to define its Lebesgue measure m . Therefore, to prove the corollary it is enough to put $f|_{A(K)}: A(K) \rightarrow A(K)$ in the role of $f: M \rightarrow M$ and $m|_{A(K)}$ in the role of m , where $m|_{A(K)} := m(B \cap A(K))$ for any Borel set $B \subset M$.

3 Existence of statistical attractors

In this section we prove Theorem 2.1 stating the existence of statistical attractors. For further uses we introduce in this section some definitions which impose additional minimality conditions to the statistical attractors (Definitions 3.1 and 3.2). At the end of this section we restate the result of existence under those additional conditions (Theorems 3.3 and 3.4).

Proof of Theorem 2.1

Let us fix $0 < \alpha \leq 1$. Consider the family \aleph_α of all the α -obs statistical attractors (non necessarily minimal). The family \aleph_α is nonempty since it trivially contains the manifold M .

Define in \aleph_α the partial order $K_1 \leq K_2$ if $K_1 \subset K_2$. Since the attractors are all non empty compact sets, any chain $\{K_i\}_{i \geq 1}$, $K_{i+1} \leq K_i \forall i \geq 1$ in \aleph_α has a non empty compact intersection:

$$K := \bigcap_{i=1}^{+\infty} K_i.$$

Let us prove that $K \in \aleph_\alpha$. First, since $K_i = f^{-1}(K_i)$ for all $i \geq 1$, then $K = \bigcap_{i \geq 1} f^{-1}(K_i) = f^{-1}(K)$. Then, K is also f -invariant. Second, to deduce that $K \in \aleph_\alpha$ it is enough to prove that $m(B(K)) \geq \alpha$, where m is the Lebesgue measure and $B(K)$ is the basin of statistical attraction of K constructed in Definition 1.1.

For any $\epsilon > 0$ and for any nonempty compact set $H \subset M$, define

$$B_\epsilon(H) := \{x \in M : \liminf_{n \rightarrow +\infty} \omega_{n,H,\epsilon}(x) > 1 - \epsilon\}, \quad \text{where} \quad (3)$$

$$\omega_{n,H,\epsilon}(x) := \frac{1}{n} \# \{0 \leq j \leq n-1 : \text{dist}(f^j(x), H) < \epsilon\} \leq 1.$$

It is standard to check that $B_{\epsilon'}(K) \subset B_{\epsilon}(K)$ if $0 < \epsilon' < \epsilon$. Therefore,

$$B(K) = \bigcap_{\epsilon > 0} B_{\epsilon}(K) = \bigcap_{n \geq 1} B_{1/n}(K),$$

and thus

$$m(B(K)) = \lim_{n \rightarrow +\infty} m(B_{1/n}(K)) = \lim_{\epsilon \rightarrow 0^+} m(B_{\epsilon}(K)).$$

So, to deduce that $m(B(K)) \geq \alpha$ it is enough to prove that $m(B_{2\epsilon}(K)) \geq \alpha$ for all $\epsilon > 0$. In fact, let us fix $\epsilon > 0$. For all $K_i \in \aleph_{\alpha}$ we have $m(B_{\epsilon}(K_i)) \geq m(B(K_i)) \geq \alpha$. Let us define the set

$$C_{\epsilon} := \bigcap_{i \geq 1} B_{\epsilon}(K_i) \subset M.$$

Since $K_{i+1} \subset K_i$, we obtain that $B_{\epsilon}(K_{i+1}) \subset B_{\epsilon}(K_i)$ for all $i \geq 1$. Therefore

$$m(C_{\epsilon}) = \lim_{i \rightarrow +\infty} m(B_{\epsilon}(K_i)) \geq \alpha > 0.$$

Now, to deduce that $m(B_{2\epsilon}(K)) \geq \alpha$ it is enough to prove that $C_{\epsilon} \subset B_{2\epsilon}(K)$. In fact, since the decreasing sequence of compacts set K_i converge to K , we have that

$$\lim_{i \rightarrow +\infty} \text{dist}(y, K_i) = \text{dist}(y, K) \quad \text{uniformly for } y \in M.$$

So, there exists $i_0 \geq 1$ such that for all $i \geq i_0$, for all $x \in C_{\epsilon}$ and for all $j \geq 0$, the following inequality holds:

$$\text{dist}(f^j(x), K) < \text{dist}(f^j(x), K_i) + \epsilon.$$

We deduce that for all $n \geq 1$, for all $x \in C_{\epsilon}$ and for all $i \geq i_0$:

$$\# \{0 \leq j \leq n-1 : \text{dist}(f^j(x), K) < 2\epsilon\} \geq \# \{0 \leq j \leq n-1 : \text{dist}(f^j(x), K_i) < \epsilon\}.$$

Since $C_{\epsilon} \subset B_{\epsilon}(K_n)$ for all $n \geq n_0$, we have that

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \# \{0 \leq j \leq n-1 : \text{dist}(f^j(x), K_i) < \epsilon\} > 1 - \epsilon \quad \forall x \in C_{\epsilon}, \quad \text{and then:}$$

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \# \{0 \leq j \leq n-1 : \text{dist}(f^j(x), K) < 2\epsilon\} \geq 1 - \epsilon > 1 - 2\epsilon \quad \forall x \in C_{\epsilon}.$$

In other words, we have obtained that $C_{\epsilon} \subset B_{2\epsilon}(K)$ for all $\epsilon > 0$, as wanted. So we have proved that $K \in \aleph_{\alpha}$. We conclude that any chain in the partially ordered set \aleph_{α} has a minimal element. Therefore, by Zorn Lemma, there exist minimal elements in \aleph_{α} . This means that there exist α -obs statistical attractors $K \subset M$, that do not

contains proper subsets that are also α -obs statistical attractors. So, the existence of minimal α -obs statistical attractors is proved for any previously specified value of $\alpha \in (0, 1]$.

To end the proof of Theorem 2.1 it is left to show that the minimal 1-obs. statistical attractor is unique. In fact, it is straightforward to check that the basins $B(K_1)$ and $B(K_2)$ of two statistical attractors K_1 and K_2 satisfy $B(K_1 \cap K_2) = B(K_1) \cap B(K_2)$. If K_1 and K_2 are 1-obs., then $m(B(K_{1,2})) = 1$ and so $K_1 \cap K_2$ is also a 1-obs. statistical attractor. Finally, if besides K_1 and K_2 are minimal 1-obs. we conclude that $K_1 \cap K_2 = K_1 = K_2$ as wanted. \square

For further uses we introduce the following definitions:

Definition 3.1 Let $0 \leq \alpha \leq 1$ and let $M' \subset M$ be a Borel set such that $M' \subset f^{-1}(M')$ and $m(M') \geq \alpha$. We say that a nonempty, compact and f -invariant set $K \subset M$ is an α -obs statistical attractor restricted to M' , if its basin $B(K)$, as defined in 1.1, satisfies:

$$m(B(K) \cap M') \geq \alpha. \quad (4)$$

We say that an α -obs statistical attractor is *minimal restricted to M'* if it satisfies the inequality (4) and has no proper, nonempty and compact subsets that satisfy it.

Definition 3.2 Let $B \subset M$ be a Borel set such that $B \subset f^{-1}(B)$ and $m(B) \geq \alpha > 0$. We say that a nonempty compact and f -invariant set $K \subset M$ is a *statistical attractor attracting B* if its basin of attraction $B(K)$, as defined in 1.1, satisfies:

$$B(K) \supset B \text{ } m - \text{a.e.} \quad \text{In other words, } m(B \setminus B(K)) = 0. \quad (5)$$

Since $m(B) \geq \alpha$ any statistical attractor attracting B is α -obs.

We say that a statistical attractor is *minimal attracting B* if it satisfies the condition (5) and has not proper, nonempty and compact subsets that satisfy it.

It is standard to check that an α -obs minimal statistical attractor K is also α -obs minimal restricted to its basin, and minimal attracting its basin. Nevertheless, the converse assertions are not (a priori) necessarily true.

Theorem 3.3 Let $M' \subset M$ be a Borel set such that $M' \subset f^{-1}(M')$ and $m(M') \geq \alpha > 0$. Then there exists an α -obs statistical attractor that is minimal restricted to M' , according to Definition 3.1.

Proof. Apply the same proof of Theorem 2.1, but using M' instead of M , $B(K) \cap M'$ instead of $B(K)$ and $B_\epsilon(H) \cap M'$ instead of $B_\epsilon(H)$. \square

Theorem 3.4 Let $B \subset M$ be a Borel set such that $B \subset f^{-1}(B)$ and $m(B) > 0$. Then, there exists a statistical attractor that is minimal attracting B , according to Definition 3.2.

Proof. Apply the same proof of Theorem 2.1, but defining the family \aleph_B (instead of \aleph_α) of all the statistical attractors $K \subset M$ such that $B(K) \supset B$ m -a.e. \square

In Theorem 4.5 we will show how to construct a statistical attractor that is minimal attracting B , being $B \subset M$ any forward invariant set with positive Lebesgue measure.

4 Probabilistic characterization of statistical attractors by the Krylov-Bogolyubov procedure

In this section we prove Theorem 2.2. This result characterizes any α -obs minimal statistical attractor as the compact support of a set of SRB-like measures. For a seek of completeness, before proving Theorem 2.2 we revisit the concept of SRB-like measures. We do this in the following Definition 4.1. The notion was taken from [18] and is a generalization of the observable measures defined in [20], and of the asymptotic measures respect to Lebesgue defined in [21]. We notice that the notions introduced in [20] and [21] are not equivalent to Definition 4.1, but are stronger. In fact, for instance the observable measures defined in [20] do not necessarily exist.

Previously to Definition 4.1, let us fix a metric dist^* inducing the weak* topology in the space \mathcal{M} of all the Borel probability measures on M .

Definition 4.1 (SRB-LIKE OR PHYSICAL-LIKE MEASURES)

Let $B \subset M$ be a forward invariant set (i.e. $B \subset f^{-1}(B)$) that has positive Lebesgue measure. We say that a probability measure μ is *SRB-like or physical-like* for $f|_B$, if for all $\epsilon > 0$ the following set $B_\epsilon(\mu) \subset B$ has positive Lebesgue measure:

$$B_\epsilon(\mu) := \{x \in B : \text{dist}^*(\mathcal{L}^*(x), \mu) < \epsilon\},$$

where $\mathcal{L}^*(x)$ is the nonempty weak*-compact set defined in (2).

We call $B_\epsilon(\mu)$ *the basin of ϵ -weak statistical attraction* of the probability μ .

To justify the name “SRB-like or physical-like measure”, compare Definition 4.1 with Definition 6.2 of SRB or physical measures.

We denote $\mathcal{O}_{f|B}$ to the family of all the SEB-like measures μ for $f|_B$.

We notice that if f is continuous, then all the measures in $\mathcal{O}_{f|B}$ are f -invariant. In other words $\mathcal{O}_{f|B} \subset \mathcal{M}_f$. In fact, $\mathcal{L}^*(x) \subset \mathcal{M}_f$ for all $x \in M$ and under the additional assumption of continuity of f the set \mathcal{M}_f of f -invariant Borel probabilities is non-empty and weak*-closed.

The following lemmas 4.2 and 4.3 are reformulations of some of the results communicated in [18]. For a seek of completeness we reproduce their proofs in this paper.

Lemma 4.2 $\mathcal{O}_{f|B}$ is weak*-compact and nonempty.

Proof. It is immediate that $\mathcal{O}_{f|B}$ is weak*-compact, because it is closed in the space \mathcal{M} , which is a compact metric space for any metric inducing its weak* topology. Let us prove that it is nonempty. Assume by contradiction that no measure in \mathcal{M} is SRB-like. Then for all $\mu \in \mathcal{M}$ there exists $\epsilon > 0$ such that $m(B_\epsilon(\mu)) = 0$, where m denotes the Lebesgue measure on M . Since \mathcal{M} is compact, there exists a finite covering of \mathcal{M} with balls $\{\mathcal{B}_i\}_{i=1,\dots,m}$ of radii $\epsilon_i : i = 1, \dots, m$ and centered at $\mu_i : i = 1, \dots, m$, such that $m(B_{\epsilon_i}(\mu_i)) = 0$ for all $i = 1, \dots, m$. Since $m(\bigcup_{i=1}^m B_{\epsilon_i}(\mu_i)) = 0$ and $\bigcup_{i=1}^m B_{\epsilon_i}(\mu_i) \supset \{x \in B(K) : \mathcal{L}^*(x) \cap \mathcal{M} \neq \emptyset\}$, we conclude that for Lebesgue almost all $x \in B(K)$ the limit set $\mathcal{L}^*(x)$ (which by definition is always contained in the space \mathcal{M}), is empty. This is a contradiction since the space \mathcal{M} is sequentially compact when endowed with the weak* topology, and thus, $\mathcal{L}^*(x) \neq \emptyset$ for all $x \in B(K)$. \square

Lemma 4.3 The set $\mathcal{O}_{f|B}$ is the minimal weak* compact set in the space \mathcal{M} of Borel probabilities such that $\mathcal{L}^*(x) \subset \mathcal{O}_{f|B}$ for Lebesgue almost all $x \in B$.

Proof. Let us first prove that for m -almost all $x \in B$ the limit set $\mathcal{L}^*(x)$ is contained in $\mathcal{O}_{f|B}$. Assume by contradiction that the set of such points x has m -measure smaller than $m(B)$. Then $\lim_{\epsilon \rightarrow 0} m(A_\epsilon < m(B))$, where

$$A_\epsilon := \{x \in B : \max\{\text{dist}^*(\nu, \mu) : \nu \in \mathcal{L}^*(x), \mu \in \mathcal{O}_{f|B}\} < \epsilon\}.$$

So, for some $\epsilon_0 > 0$ small enough $m(B \setminus A_{\epsilon_0}) > 0$. In other words, for a Lebesgue positive set of points $x \in B$, the limit set $\mathcal{L}^*(x)$ intersects the weak*-compact set $\mathcal{K} := \{\mu \in \mathcal{M} : \text{dist}^*(\mu, \mathcal{O}_{f|B}) \geq \epsilon_0\}$. Therefore, at least one of the measures $\mu \in \mathcal{K}$ satisfies $m(B_\epsilon(\mu)) > 0$ for all $0 < \epsilon \leq \epsilon_0$, where

$$B_\epsilon(\mu) := \{x \in B : \text{dist}^*(\mathcal{L}^*(x), \mu) < \epsilon\}.$$

In fact, if the latter assertion were not true, we would cover \mathcal{K} with a finite number of balls $\{\mathcal{B}_i\}_{i=1,\dots,m}$ such that for Lebesgue almost all point $x \in B$, $\mathcal{L}^*(x) \cap \mathcal{B}_i = \emptyset$ for all $i = 1, \dots, m$. Thus $\mathcal{L}^*(x) \cap \mathcal{K} = \emptyset$ for Lebesgue almost all $x \in B$, contradicting the construction of the set \mathcal{K} .

Thus, there exists $\mu \in \mathcal{K}$ such that $m(B_\epsilon(\mu)) > 0$ for all $0 < \epsilon \leq \epsilon_0$. Then, after Definition 4.1 the probability measure μ is SRB-like for $f|B$. Therefore $\mathcal{K} \cap \mathcal{O}_{f|B} \neq \emptyset$, contradicting the construction of the compact set \mathcal{K} . This ends the proof of the first assertion: for m -almost all $x \in B$, $\mathcal{L}^*(x) \subset \mathcal{O}_{f|B}$.

Second, let us prove that $\mathcal{O}_{f|B}$ is minimal among all the weak* compact sets containing $\mathcal{L}^*(x)$ for Lebesgue almost all $x \in B$. In fact, if $\emptyset \neq \mathcal{K} \subset \mathcal{O}_{f|B}$, and \mathcal{K} is compact, any measure $\mu \in \mathcal{O}_{f|B}$ is at positive weak*-distance, say $\epsilon > 0$ from

\mathcal{K} . After Definition 4.1 there exist a m -positive set of points $x \in B$ such that $\text{dist}^*(\mathcal{L}^*(x), \mu) < \epsilon$. Therefore for those points $\mathcal{L}^*(x) \not\subset \mathcal{K}$. We conclude that $\mathcal{O}_{f|B}$ has no nonempty, proper and compact subset containing $\mathcal{L}^*(x)$ for Lebesgue almost all $x \in B$. This ends the proof that $\mathcal{O}_{f|B}$ is minimal with such a property. \square

Lemma 4.4 *If K is a compact set such that $\mu(K) = 1$ for all $\mu \in \mathcal{O}_{f|B}$, then K is a statistical attractor whose basin $B(K)$ contains B .*

Proof. Fix $\epsilon > 0$ and choose any continuous function $\psi \in C^0(M, [0, 1])$ such that $\psi|_K = 1$ and $\psi(y) = 0$ for all $y \in M$ such that $\text{dist}(y, K) \geq \epsilon$. Choose and fix $x \in B$, and a sequence of natural numbers $n_i \rightarrow +\infty$ such that the following limits exist:

$$L = \lim_{i \rightarrow +\infty} \frac{1}{n_i} \# \{0 \leq j \leq n_i - 1 : \text{dist}(f^j(x), K) < \epsilon\}.$$

$$\mu = \lim_{i \rightarrow +\infty}^* (1/n_i) \sum_{j=0}^{n_i-1} \delta_{f^j(x)} \in \mathcal{L}^*(x).$$

On the one hand, $\mathcal{L}^*(x) \subset \mathcal{O}_{f|B}$ for m -a.e. $x \in B$ and $\mu(K) = 1$ for all $\mu \in \mathcal{O}_{f|B}$. Therefore, the expected value of ψ respect to the probability μ is equal to 1. In fact: $1 \leq \int \psi d\mu \leq \int_K \psi d\mu = \mu(K) = 1$. On the other hand, the limit* in the space of probabilities can be computed as follows:

$$1 = \int \psi d\mu = \lim_{i \rightarrow +\infty} \int \psi d \left(\frac{1}{n_i} \sum_{j=0}^{n_i-1} \delta_{f^j(x)} \right) = \lim_{i \rightarrow +\infty} \frac{1}{n_i} \sum_{j=0}^{n_i-1} \psi(f^j(x)).$$

We deduce that $\lim_{i \rightarrow +\infty} (1/n_i) \sum_{j=0}^{n_i-1} \psi(f^j(x)) = 1$. By construction of the function ψ :

$$(1/n) \sum_{j=0}^{n-1} \psi(f^j(x)) \leq (1/n) \# \{0 \leq j \leq n-1 : \text{dist}(f^j(x), K) < \epsilon\}.$$

Then:

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \# \{0 \leq j \leq n-1 : \text{dist}(f^j(x), K) < \epsilon\} = 1 \quad m - \text{a.e. } x \in B.$$

We deduce that $x \in B(K)$ for Lebesgue almost all $x \in B$, and so K is a Milnor's attractor whose basin contains Lebesgue a.e. B . \square

End of the proof of Theorem 2.2.

Consider the basin of attraction $B(K)$ of the statistical attractor K . By hypothesis $m(B(K)) \geq \alpha > 0$. It is straightforward to check that if $x \in B(K)$ then

$f(x) \in B(K)$ (even if f is only a measurable map that is not continuous). Then, we can apply Definition 4.1, and consider the set $\mathcal{O}_{f|B(K)}$ of all the SRB-like measures for $f|_{B(K)}$. After Lemmas 4.2 and 4.3 (denoting $\mathcal{O}_f(K)$ to $\mathcal{O}_{f|B(K)}$), assertions (a) and (b) of Theorem 2.2 are proved.

Now, let us prove the two assertions in (c). By hypothesis K is a Milnor-attracting minimal attracting $B(K)$ or α -obs minimal restricted to $B(K)$. To deduce that K is the compact support of all the probabilities in $\mathcal{O}_f(K)$, we will first prove that $\mu(K) = 1$ for all $\mu \in \mathcal{O}_f(K)$, and then prove that K is the minimal compact set in M that has such property.

Fix $\mu \in \mathcal{O}_f(M)$. Choose an arbitrarily small $\epsilon > 0$ and denote

$$V_\epsilon = \{x \in M : \text{dist}(x, K) < \epsilon\}.$$

Construct a continuous real function $\psi \in C^0(M, [0, 1])$ such that $\psi|_K = 1$ and $\psi(x) = 0$ if $x \notin V_\epsilon$. After the continuity of ψ there exists $0 < \epsilon' < \epsilon$ such that $\psi(x) > 1 - \epsilon \ \forall x \in V_{\epsilon'}(K)$. Let us compute the expected value of ψ respect to the probability μ :

$$\int \psi d\mu = \int_{V_\epsilon} \psi d\mu \leq \mu(V_\epsilon). \quad (6)$$

Recall Equality (2) which defines $\mathcal{L}^*(x)$ for all $x \in M$, Definition 1.1 of the basin $B(K)$ of the statistical attractor K , and Equality (3) defining the set $B_\epsilon(K) \subset M$ for all $\epsilon > 0$. Then $B(K) = \bigcap_{\epsilon > 0} B_\epsilon(K)$. Applying the statements (a) and (b), and the definition of SRB-like measure μ , there exists $x \in B(K) \subset B_{\epsilon'}(K)$ and $\tilde{\mu} \in \mathcal{L}^*(x)$ such that $|\int \psi d\mu - \int \psi d\tilde{\mu}| < \epsilon$. Therefore, there exists a subsequence $\{\nu_{n_i}(x)\}_{i \geq 1}$ convergent to $\tilde{\mu}$ in the weak* topology of \mathcal{M} , where $\nu_n(x) := (1/n) \sum_{j=0}^{n-1} \delta_{f^j(x)}$ for all $n \geq 1$. Thus:

$$\begin{aligned} \int \psi d\tilde{\mu} &= \lim_{i \rightarrow +\infty} \int \psi d \left(\frac{1}{n_i} \sum_{j=0}^{n_i-1} \delta_{f^j(x)} \right) = \lim_{i \rightarrow +\infty} \frac{1}{n_i} \sum_{j=0}^{n_i-1} \psi(f^j(x)) \geq \\ &(1 - \epsilon) \lim_{i \rightarrow +\infty} \frac{1}{n_j} \#\{0 \leq j \leq n_j - 1 : f^j(x) \in V_{\epsilon'}\}. \end{aligned}$$

Since $x \in B_{\epsilon'}(K)$ the limit of the latter term above, is larger than $1 - \epsilon'$ (recall Equality (3)). Therefore $\int \psi d\tilde{\mu} \geq (1 - \epsilon)(1 - \epsilon') \geq (1 - \epsilon)^2$ and thus $\int \psi d\mu \geq (1 - \epsilon)^2 - \epsilon$. Joining with Inequality (6), we deduce: $\mu(V_\epsilon) \geq (1 - \epsilon)^2 - \epsilon \ \forall \epsilon > 0$. Taking $\epsilon \rightarrow 0^+$ and since the compact set K is the decreasing intersection of the open sets V_ϵ , we obtain:

$$1 \geq \mu(K) = \lim_{\epsilon \rightarrow 0^+} \mu(V_\epsilon) \geq \lim_{\epsilon \rightarrow 0^+} (1 - \epsilon)^2 - \epsilon = 1.$$

So, we have proved that $\mu(K) = 1$ for all $\mu \in \mathcal{O}_f(K)$, as wanted.

Finally, it is left to prove that if $K' \subset K$ is a nonempty compact set such that $K \setminus K' \neq \emptyset$, then $\mu(K') < 1$ for some $\mu \in \mathcal{O}_f(K)$. By hypothesis K is a statistical attractor minimal attracting $B(K)$ or α -obs minimal restricted to $B(K)$. Therefore, after Definitions 3.2 and 3.1, the set $B(K')$ (defined as in 1.1), excludes a Lebesgue-positive set of points of $B(K)$, namely there exists a positive Lebesgue measure set $C = B(K) \setminus B(K')$. Then: $C = \bigcup_{\epsilon > 0} B(K) \setminus B_\epsilon(K') \subset B(K)$, $m(C) > 0$, where $B_\epsilon(K')$ is defined in Equality (3). Fix a point $x \in C$ and fix $\epsilon > 0$ such that $x \notin B_\epsilon(K')$. Choose a continuous real function $\psi \in C^0(M, [0, 1])$ such that $\psi|_{K'} = 1$ and $\psi(y) = 0$ for all y such that $\text{dist}(y, K') \geq \epsilon$. After Equalities (3) we have, for all $x \in C$, $\liminf_{N \rightarrow +\infty} \omega_{\epsilon, N}(x, K', \epsilon) \leq 1 - \epsilon$. In other words, there exists a sequence $n_i \rightarrow +\infty$ such that

$$\lim_{i \rightarrow +\infty} \frac{1}{n_i} \# \{0 \leq j \leq n_i - 1 : \text{dist}(f^j(x), K') < \epsilon\} \leq 1 - \epsilon.$$

Therefore,

$$\begin{aligned} \limsup_{i \rightarrow +\infty} \int \psi d\nu_{n_i}(x) &:= \limsup_{i \rightarrow +\infty} \int \psi d \left(\frac{1}{n_i} \sum_{j=1}^{n_i} \delta_{f^j(x)} \right) = \\ &= \limsup_{i \rightarrow +\infty} \frac{1}{n_i} \sum_{j=0}^{n_i-1} \psi(f^j(x)) \leq \\ &\leq \limsup_{i \rightarrow +\infty} \frac{1}{n_i} \# \{0 \leq j \leq n_i - 1 : \text{dist}(f^j(x), K') < \epsilon\} \leq 1 - \epsilon. \end{aligned}$$

Taking if necessary a subsequence of $\{n_i\}_{i \geq 0}$ (which we still denote $\{n_i\}_{i \geq 0}$) such that $\{\nu_{n_i}(x)\}_{i \geq 0}$ is convergent in the weak* topology to a probability measure $\mu \in \mathcal{L}^*(x)$, we obtain: $\int \psi d\mu = \lim_{i \rightarrow +\infty} \int \psi d\nu_{n_i}(x) \leq 1 - \epsilon < 1$. But, on the other hand, $\int \psi d\mu \geq \int_{K'} \psi d\mu = \mu(K')$. So $\mu(K') < 1$.

We have proved that for all $x \in C$ there exists a measure $\mu = \mu_x \in \mathcal{L}^*(x)$ such that $\mu_x(K') < 1$. Recall that $C \subset B(K)$ and $m(C) > 0$. After the statement (a) of Theorem 2.2 (which we have already proved), $\mathcal{L}^*(x) \subset \mathcal{O}_f(K)$ for Lebesgue almost all points $x \in B(K)$, so in particular for Lebesgue almost all points $x \in C$. So, $\mu(K') < 1$ for some $\mu \in \mathcal{O}_f(K)$, as wanted. \square

Theorem 3.4 states that, for any given forward invariant set B with positive Lebesgue measure, there exists a statistical attractor that is minimal attracting B . We will show how this attractor can be constructed:

Theorem 4.5 *Let $B \subset M$ be a nonempty and forward invariant set (i.e. $B \subset f^{-1}(B)$) such that $m(B) > 0$. Construct the set $\mathcal{O}_{f|B}$ of all the SRB-like measures of $f|_B$. Then, the minimum compact set $K \subset M$ such that $\mu(K) = 1$ for all $\mu \in \mathcal{O}_{f|B}$ is a statistical attractor, its basin of attraction contains B , and K is minimal attracting B .*

Proof. After Theorem 3.4 there exists a statistical attractor K' that is minimal attracting B , i.e. $B(K') \supset B$. It is enough to prove that $K' = K$. Applying Lemma 4.4, and Definition 3.2 we have $K' \subset K$. On the other hand, applying the assertions (a) and (b) of Theorem 2.2 combined with Lemma 4.3, the set $\mathcal{O}_f(K')$ of probability measures in Theorem 2.2 coincides with the set $\mathcal{O}|_{f|B(K')}$ in Lemma 4.3. After the assertion (c) of Theorem 2.2 and the minimality of the Milnor attractor K' , we deduce that K' is the common compact support of all the measures in $\mathcal{O}|_{f|B(K')}$. As $B(K') \subset B$, then $\mathcal{O}|_{f|B} \subset \mathcal{O}|_{f|B(K')}$. So, $K \subset K'$. We conclude that $K' = K$, as wanted. \square

5 Decomposition of the space

In this section we prove Theorem 2.3. This result states the covering of Lebesgue almost all the space with the basins of attraction of statistical attractors satisfying an observable minimality condition.

Proof of Theorem 2.3.

After Theorem 2.1, there exists an α -minimal observable statistical attractor K_1 . Then $m(B(K_1)) \geq \alpha$. Denote $\alpha_1 = \alpha$.

1ST. STEP.

Denote $r_1 = m(B(K_1)) \geq \alpha$. Either $r_1 = 1$ or $1 - \alpha < r_1 < 1$ or $\alpha \leq r_1 \leq 1 - \alpha$.

(1) In the first case, Theorem 2.3 becomes trivially proved with $p = 1$.

(2) In the second case denote $\alpha'_2 = 1 - m(B(K_1))$. Then $0 < \alpha'_2 < \alpha$. Consider the set $B := M \setminus B(K_1)$. After Definition 1.1 it is standard to check that $f^{-1}(B(K_1)) = B(K_1)$. Therefore $f^{-1}(B) = B$. After Theorem 3.4 there exists a statistical attractor K_2 minimal attracting B , namely $B \subset B(K_2)$. As $\alpha'_2 = m(B)$, the attractor K_2 is α'_2 -obs minimal restricted to B . If $m(B(K_2)) \geq \alpha$ then K_2 is α -obs. minimal restricted to B (because it is α'_2 -obs minimal restricted to B and $\alpha'_2 < \alpha$.) On the other hand, if $m(B(K_2)) < \alpha$, then K_2 is trivially α'_2 -obs minimal respect to B . Therefore, in this second case Theorem 2.3 is proved with $p = 2$ taking $\alpha_2 := \alpha$ (if $m(B(K_2)) \geq \alpha$) or $\alpha_2 := \alpha'_2$ (if $m(B(K_2)) < \alpha$).

(3) In the third case, the set $B := M \setminus B(K_1)$ has Lebesgue measure $m(B) = \alpha'_2 \geq \alpha$. After Theorem 3.3 there exists an α -obs statistical attractor K_2 that is minimal restricted to B and we go to the second step, after discussing again into three sub-cases

2ND. STEP.

Denote $r_2 := m(B(K_1) \cup B(K_2)) = m(B(K_1)) + m(B(K_2) \setminus B(K_1)) \geq 2\alpha$.

Either $r_2 = 1$, or $1 - \alpha < r_2 < 1$ or $2\alpha \leq r_2 \leq 1 - \alpha$.

(1') In the first case, Theorem 2.3 becomes trivially proved with $p = 2$.

(2') In the second case, Theorem 2.3 becomes proved with $p = 3$, after the construction of a statistical attractor K_3 following the same arguments that were used in (2) to construct K_2 .

(3') In the third case, we can construct an α -obs statistical attractor K_3 applying the same arguments that we used above in (3) to construct K_2 , and go to the following step, after a discussion about the value of $r_3 := m(B(K_1) \cup B(K_2) \cup B(K_3))$.

LAST STEP.

After $p \geq 1$ steps as above, define the number

$$r_p = m\left(\bigcup_{i=1}^p B(K_i)\right) = \sum_{i=1}^p m(B(K_i) \setminus \bigcup_{j=1}^{i-1} B(K_j)) \geq p \alpha.$$

Since $r_p \leq 1$, the last step p satisfies $p \leq 1/\alpha$ and $1 - r_p < \alpha$. So, $p = \text{Integer part}(1/\alpha)$. Therefore, in the last step p we always eventually drop in the cases similar to (1) or (2). We conclude that either there exist a finite number p of α -obs statistical attractors satisfying the assertions of Theorem 2.3, or there exist a finite number $p + 1$ of statistical attractors, also satisfying the assertions of this theorem, being all α -obs, except at most one one which is α_p -obs with $\alpha_p = 1 - r_p < \alpha$. □

6 Milnor's attractors and Pugh-Shub ergodic attractors.

In this section we recall two old definitions in which the statements and proofs of some results of this paper were inspired, besides the Ilyashenko's definition of attractor.

Definition 6.1 (MILNOR'S ATTRACTOR)

Let $f: M \rightarrow M$ be a Borel measurable map on a compact manifold M . A compact set $K \subset M$ is a *Milnor's attractor* if the set $A(K) \subset M$ of all the initial states $x \in M$ such that the omega-limit set $\omega(x)$ is contained in K , has positive Lebesgue measure.

We recall that $\omega(x)$ is the compact nonempty set in M composed by the limits of all the convergent subsequences of the orbit $\{f^n(x)\}_{n \in \mathbb{N}}$. We call $A(K)$ *the basin of topological attraction* of K . We say that a Milnor's attractor is α -observable if $m(A(K)) \geq \alpha$, where m denotes the Lebesgue measure.

Note: Since usually $\omega(x)$ does not depend continuously on x , the basin $A(K)$ of a Milnor's attractor may contain no neighborhood of K . Even more, the basins of attraction of two different Milnor's attractors may be topologically riddling or intermingled (see for instance [22]).

Definition 6.2 (SRB OR PHYSICAL MEASURES AND ERGODIC ATTRACTORS)

A probability Borel measure μ on M is called *SRB or physical* if the set

$$B(\mu) := \{x \in M : \lim_{n \rightarrow +\infty}^* \nu_n(x) = \mu\} = \{x \in M : \mathcal{L}^*(x) = \{\mu\}\}$$

has positive Lebesgue measure.

In the definition above \lim^* denotes the limit in the space \mathcal{M} of all the Borel probabilities on M , endowed with the weak* topology, $\sigma_n(x)$ denotes the empirical probability defined in Equality (1) and $\mathcal{L}^*(x)$ denotes the limit set in \mathcal{M} of the sequence of empirical probabilities, defined in Equality (2).

We call $B(\mu)$ the *basin of statistical attraction* of μ .

We call *ergodic attractor* to any nonempty, compact and f -invariant set $K \subset M$ (i.e. $f^{-1}(K) = K$) that is the compact support of a physical measure μ (i.e. K is the minimal compact set such that $\mu(K) = 1$).

Note: Since usually $\mathcal{L}^*(x)$ does not depend continuously on x , the basin $B(K)$ of an ergodic attractor K may contain no neighborhood of K . Even more, the basins of attraction of two different ergodic attractors may be topologically riddling or intermingled (see for instance 11.1.1 in the book [23]).

After Definition 6.2, any physical measure is f -invariant provided that f is continuous. Nevertheless, physical measures are not necessarily ergodic. (See for instance Bowen Example 7.2.)

We notice that the definition of physical measure also holds for any Borel measurable $f: M \rightarrow M$, but in this case μ is not necessarily f -invariant. For instance $f: [0, 1] \rightarrow [0, 1]$ defined by $f(0) = 1$, $f(x) = (1/2)x$ for all $x \neq 0$, has the physical measure δ_0 (whose basin is $[0, 1]$), which is not f -invariant.

7 Examples

In this section we revisit some paradigmatic examples, which are mostly well known since a long time ago. They show that Definition 1.1, as well as the abstract results stated in Section 2, are adequate generalizations of those concerning to Milnor's attractors and Pugh-Shub ergodic attractors.

First, to illustrate the main non trivial difference between Milnor's attractors and ergodic attractors, we recall the following case:

Example 7.1 (HU-YOUNG DIFFEOMORPHISM)

Consider the topologically transitive C^2 diffeomorphism f studied in [24]: it acts in the 2-torus \mathbf{T}^2 , and is obtained by an isotopy from a linear Anosov in such a way that the eigenvalues of df at a fixed point x_0 are modified. Along the contracting subspace the eigenvalue is still smaller than 1, while in the eigendirection tangent to a

topologically expansive (topologically unstable) C^1 submanifold, the eigenvalue was weakened to become equal to 1. In [24] it is proved that, for such f , the sequence in Equality (1) of empiric probabilities converges to δ_{x_0} in the space \mathcal{M} of all the Borel probabilities (endowed with the weak*-topology), for Lebesgue a.e. $x \in \mathbf{T}^2$. In other words, δ_{x_0} is a physical measure, the ergodic attractor is $K = \{x_0\}$ and its basin of attraction covers \mathbf{T}^2 up to a set of Lebesgue measure zero. Therefore, the frequency of visits to any arbitrarily small neighborhood of the fixed point is asymptotically equal to 1, for Lebesgue almost all the initial state. Thus, the asymptotic frequency of visits to all the rest of the space is zero. Nevertheless, since f is transitive, the unique (and thus minimal) Milnor's attractor is the whole torus. In this example the unique ergodic attractor $\{x_0\}$ is not a Milnor's attractor because it is a saddle, with a Lebesgue zero, one-dimensional and C^1 -immersed stable manifold $W^s(x_0) \subset M$.

Inspired both in the Milnor and in the ergodic attractors, we have introduced Definition 1.1, which is an equivalent restatement of the Iliashenko's statistical attractor. Any Milnor's attractor and any ergodic attractor are also statistical attractors. Nevertheless, the converse assertion is false. In fact, we will see in case (C) of the following Example 7.2, that there exists a statistical attractor that is not a Milnor's attractor nor an ergodic attractor.

Example 7.2 (BOWEN HOMEOMORPHISM)

This example is attributed to Bowen in [25] and [26], and was also posed in [27]. Consider in a two dimensional manifold a non singular homeomorphism f (namely $m(f^{-1}(B)) = 0$ if and only if $m(B) = 0$, where m is the Lebesgue measure). Construct such an f so that:

- (i) f has three fixed points x_1, x_2 and x_3 .
- (ii) When restricted to the union of three compact pairwise disjoint neighborhoods U_1, U_2 and U_3 of x_1, x_2 and x_3 respectively, f is a diffeo onto $f(U_1 \cup U_2 \cup U_3)$, and the fixed points x_1 and x_2 are hyperbolic saddles, while x_3 is a hyperbolic source.
- (iii) $W_1^s \setminus \{x_1\} = W_2^u \setminus \{x_2\}$, $W_1^u \setminus \{x_1\} = W_2^s \setminus \{x_2\}$. We denote $W_{1,2}^{s,u}$ to half-branches of the global one-dimensional stable and unstable manifolds of $x_{1,2}$ respectively. They are embedded topological arcs, that are besides of C^1 type in a neighborhood of the saddles $x_{1,2}$. So $W_1^s \cup W_2^s$ is a compact, simple and closed arc which is the boundary of an open set V homeomorphic to a 2-ball.
- (iv) All the orbits in $V \setminus \{x_3\}$ include x_1 and x_2 in their ω -limit and have $\{x_3\}$ as α -limit set.

Note that such a C^0 map f can be constructed for any previously specified values of the eigenvalues of df at the two saddles x_1 and x_2 , and after an adequate choice of the values $f(x)$ for $x \in V \setminus (U_1 \cup U_2 \cup U_3)$.

Let us consider the restricted dynamical system $f|_{\overline{V}}$. On the one hand and from the topological viewpoint, all the orbits of $V \setminus \{x_3\}$ are attracted to (i.e. have ω -limit set contained in) the boundary ∂V . This closed arc is the unique 1-obs. minimal

Milnor's attractor of $f|_V$. On the other hand, from the statistical viewpoint the behavior of the system is much more delicate (i.e. when looking the asymptotic behavior of the sequence of empirical probability measures defined in Equality (1)). In fact, necessarily one and only one of the following properties (A), (B) or (C) holds, and any of the three is realizable if the eigenvalues of x_1 and x_2 are adequately chosen and the C^0 map $f|_{V \setminus (U_1 \cup U_2)}$ is well constructed:

(A) There exists a unique SRB or physical measure attracting $V \setminus \{x_3\}$ which is δ_{x_1} or δ_{x_2} . In this case either $\{x_1\}$ or $\{x_2\}$ is an ergodic attractor, it is the unique statistical attractor and the physical measure δ_{x_i} is ergodic. We prove that this case is nonempty (see the argument following the end of Example 7.3 in this section).

(B) There exists a unique SRB or physical measure μ attracting $V \setminus \{x_3\}$, which is $\mu = t\delta_{x_1} + (1 - t)\delta_{x_2}$ for some constant $0 < t < 1$. The existence of examples in this case (B) is stated for instance in Lemma (i) of page 457 in [27]. For the detailed construction of an example in this case, consider $\lambda = 1/\sigma$ in the Equalities of Theorem 1 of [25], and construct f such that it preserves area in both the disjoint compact neighborhoods N_1 and N_2 of the saddles, and is adequately C^0 -chosen outside $N_1 \cup N_2$ to have the two saddles in the omega-limit of all the orbits of $V \setminus \{x_3\}$. We note that after a standard computation that we sketch in the proof at the end of this section, one should construct f contracting but non hyper contractive outside $N_1 \cup N_2$, so the sequence (1) is convergent according to formulae of Theorem 1 of [25] with the parameters $\lambda = 1/\sigma$. Therefore, in this case (B), the set $\{x_1, x_2\}$ is an ergodic attractor, it is the unique statistical attractor, and the physical measure μ is non ergodic. Moreover, for an adequate choice of the eigenvalues of x_1 and x_2 one can obtain this property for any previously specified value of $t \in (0, 1)$ (see the proof at the end of this section).

(C) There does not exist any physical measure, since for Lebesgue almost all the points $x \in V$, the limit set $\mathcal{L}^*(x)$ of the empirical distributions of Equality (2) is a segment in the space \mathcal{M} of probabilities. In other words, the sequence (1) of empiric probabilities for $f|_{\overline{V}}$ does not converge for Lebesgue a.e. initial state. Thus, there does not exist any ergodic attractor. The existence of C^2 examples in this case (C) is proved in [25] and [26] for which the set $\mathcal{O}_{f|_{\overline{V}}}$ of SRB-like measures is a segment which is always properly contained in $[\delta_{x_1}, \delta_{x_2}] \subset \mathcal{M}$. Nevertheless, one can construct f of C^0 class in \overline{V} such that the set of SRB-like measures for $f|_{\overline{V}}$ is exactly the segment $[\delta_{x_1}, \delta_{x_2}]$ (see the remark at the end of this section).

In this case (C) there exist uncountably many SRB-like measures for $f|_{\overline{V}}$ (after Theorem 1.7 of [18]). All of them are supported on $\{x_1, x_2\}$, due to the Poincaré Recurrence Theorem. After Theorem 4.5 the set $\{x_1, x_2\}$ is a statistical attractor. Besides, since the common minimal compact support of all the measures in $\mathcal{L}^*(x)$ is $\{x_1, x_2\}$ for Lebesgue a.e. $x \in V$, this statistical attractor is the unique α -obs

minimal one, for all $0 < \alpha \leq 1$. In other words, in this case (C) of example 7.2, the unique α -obs. minimal Milnor's attractor ∂V , and the unique α -obs. minimal statistical attractor, are different, while Pugh-Shub ergodic attractors do not exist.

Finally, let us exhibit an example that shows that if the purpose is to find the (always existing) statistical attractors of a C^1 map, even under the strong hypothesis of uniform (total or partial) hyperbolicity, then the classic approach of searching for the invariant probability measures that are absolutely continuous with respect to Lebesgue may become noneffective. On the other hand, as a consequence of Theorem 2.2, there exists an optimal description of those (always existing) probability measures that should be searched to find all the statistical attractors (see Definition 4.1). These optimal probability measures are not usually, for C^1 mappings that are not C^1 plus Hölder, those that have properties of absolute continuity with respect to Lebesgue, as the following example shows:

Example 7.3 (CAMPBELL AND QUAS EXPANDING MAPS)

Let us consider a one-dimensional, C^1 and uniformly hyperbolic map $f : S^1 \mapsto S^1$ on the circle S^1 , which is expanding, namely $|f'(x)| > 1$ for all $x \in S^1$. In Theorem 1 of [28], Campbell and Quas proved that C^1 -generically there exists a unique physical measure μ , that this measure μ is mutually singular with respect to Lebesgue, and that its basin of attraction covers Lebesgue almost all the points. This measure μ is supported on a compact subset $K \subset S^1$ (non necessarily properly contained in S^1). So, this compact support K is by definition an ergodic attractor. It is the unique statistical attractor and it is α -obs minimal for all $0 < \alpha \leq 1$, since the basin of statistical attraction of μ covers Lebesgue almost all the space. It is described by a single SRB-like measure which, in this case, is SRB.

Example 7.4 (NON ERGODIC QUAS EXPANDING MAP)

In [29] Quas gave a C^1 -non generic example, of an expanding map f on the circle S^1 (which is C^1 but non C^1 -plus-Hölder), exhibiting a statistical behavior that is rather opposite to that of the generic case of Campbell and Quas in Example 7.3. He constructed such an f preserving the Lebesgue measure m , but for which m is non ergodic. So, after Birkhoff Theorem and after the ergodic Decomposition Theorem, for m -almost every point $x \in S^1$ the set $\mathcal{L}^*(x)$ (defined in Equality (2)) consists of only one ergodic component of m . Therefore, even if also in this example there exists a unique 1-obs. minimal statistical attractor (Theorem 2.1), the set of all the SRB-like measures that describe completely its statistical behavior has more than one probability.

Proof of existence of the case (A) in Example 7.2

There exists an homeomorphism f as in Example 7.2 such that

$$\mathcal{O}|_{f|\overline{V}} = \{\delta_{x_2}\}.$$

Proof. Choose f and the eigenvalues of x_1 and x_2 so that $f|_{U_1 \cup U_2}$ is area conservative and construct first an area preserving map in a small neighborhood of ∂V . Then C^0 perturb f near ∂V , without changing $f|_{\partial V \cup U_1 \cup U_2}$, to become hyper dissipative in a small neighborhood of a fundamental domain of $W_{x_2}^s \setminus (U_1 \cup U_2)$, and hyper expanding (but not too much in relation to the hyper dissipation above) in some direction inside a small neighborhood of a fundamental domain of $W_{x_1}^s \setminus (U_1 \cup U_2)$. Precisely, construct this perturbation f such that it satisfies the following property:

At each return time $n_i(x)$ to U_2 (of any orbit with initial state $x \in V \setminus \{x_3\}$), and at each return time $n'_i(x)$ to U_1 , the distances

$$d_i(x) := \text{dist}(f^{n_i}(x), W_{x_2}^s), \quad d'_i(x) := \text{dist}(f^{n'_i}(x), W_{x_1}^s) \quad (7)$$

satisfy the inequalities

$$0 < d_{i+1}(x) < d'_i(x) - \log d'_i(x), \quad \frac{d'_i(x)}{3} \leq d'_{i+1}(x) \leq \frac{d'_i(x)}{2}.$$

(Note: it is not difficult to check that such C^0 perturbation exists. Moreover, the construction above can also be made so f is a diffeomorphism in the open set V . Nevertheless its derivative is necessarily unbounded, so f can not be constructed of C^1 type in \overline{V}).

At each visit i to the set U_2 , denote $N_i(2)$ (depending on x) to the time length that the orbit of x spends inside U_2 , and denote $N_i(1)$ to the time length that it spends inside U_1 after its i -th. visit to U_2 . Up to a constant $k > 0$, the number of iterates between the i -th. and the $(i+1)$ -th. visit to U_2 is $N_i(2) + N_i(1)$. Besides, after a standard computation, we obtain

$$N_i(2) \geq -c_2 \cdot \log_{d_i} > c_2(-\log d'_i)^2, \quad N_i(1) \leq -c_1 \cdot \log d'_i,$$

for some positive constants c_1 and c_2 . So, there $c > 0$ such that

$$N_i(2) \geq cN_i(1)^2 \quad \forall \quad i \geq 1.$$

Consider the accumulated time average $\omega_n(U_1)$ inside U_1 of the finite piece from instant 0 up to instant $n \geq 1$ of the orbit with initial state in x (namely, the relative frequency of staying in U_1).

First, if n is exactly the end instant of the staying time inside U_1 at the m -th. visit to U_1 , then $\omega_n(U_1)$ is computed as follows:

$$\begin{aligned} \omega_n(U_1) &= \frac{\sum_{i=1}^m N_i(1)}{km + \sum_{i=1}^m N_i(2) + \sum_{i=1}^m N_i(1)} \\ \frac{1}{\omega_n(U_1)} &= \frac{km + \sum_{i=1}^m N_i(2)}{\sum_{i=1}^m N_i(1)} + 1 \geq \frac{\sum_{i=1}^m [N_i(1)]^2}{\sum_{i=1}^m N_i(1)}. \end{aligned}$$

Since $N_i(1) \rightarrow +\infty$ when $i \rightarrow +\infty$, then $1/\omega_n(U_1) \rightarrow +\infty$ when $m \rightarrow +\infty$ and so $\omega_n(U_1) \rightarrow 0$.

Second, if n is larger than the end instant n' of the staying time inside U_1 at the m -th visit, but smaller than the next return time to U_1 , then $\omega_n(U_1) = (n'/n) \omega_{n'}(U_1) \leq \omega_{n'}(U_1) \rightarrow 0$ when $m \rightarrow +\infty$.

Third and finally, let us prove that also $\omega_n(U_1) \rightarrow 0$ when $m \rightarrow +\infty$, if n is any stopping time such that $f^n(x) \in U_1$ during the m -th. visit of the orbit to N_1 , but n is smaller than the end instant n' of the staying time N_m inside U_1 (at that m -th. visit). In fact $0 < n' - n < N_m \leq c(-\log d'_m)$ for some constant c which depends only of the size of U_1 and of the expanding eigenvalue of $df|_{x_1}$. Since $d'_{i+1} \geq d'_i/3$ for all $i \geq 1$, we have $d'_m \geq (1/3^m) d'_1(x)$ for all $m \geq 1$. Thus, there exists a constant $K(x) > 0$ such that $-\log d'_m \leq K(x) \cdot m$ for all $m \geq 1$. This implies that $0 < n' - n < N_m \leq c'(x) \cdot m$ where $c'(x) = c \cdot K(x)$. On the other hand $n \geq m$. Therefore,

$$\omega_n(U_1) = \frac{n'}{n} \omega_{n'}(U_1) = \omega_{n'}(U_1) \left(1 + \frac{n' - n}{n}\right) \leq \omega_{n'}(U_1) (1 + c'(x)) \rightarrow 0$$

when $m \rightarrow +\infty$.

We have proved that for all $x \in V \setminus \{x_3\}$: $\lim_{n \rightarrow +\infty} \omega_n(U_1) = 0$ (notice that when the stopping time n goes to infinity, the number m of visits to U_1 goes to infinity). On the other hand $\lim_n \omega_n(U_2) + \omega_n(U_1) = 1$. So we deduce that $\lim_n \omega_n(U_2) = 1$. Since the argument above also holds (for the same f) for an arbitrary neighborhood U'_2 of the saddle x_2 (just changing the constant k), we obtain that the sequence (1) converges to δ_2 , as wanted. \square

Remark about the case (C) of Example 7.2

There exists an homeomorphism f as in Example 7.2, for which

$$\mathcal{O}_{f|V} = [\delta_{x_1}, \delta_{x_2}].$$

Sketch of the Proof. Let us apply similar arguments to those of the proof of case (A), making f hyper dissipative near $W^s(x_2) \setminus (N_1 \cup N_2)$ but also hyper dissipative near $W^s(x_1) \setminus (N_1 \cup N_2)$. We deduce, adapting the computations in the proof of case (A), that the empirical sequence (1) will have at least two convergent subsequences, one converging to δ_{x_1} and the other to δ_{x_2} . Fix a metric dist^* in the space \mathcal{M} inducing the weak* topology. After the convex-like property stated and proved in Theorem 2.1 of [18], for all $t \in [0, 1]$ the limit set $\mathcal{L}^*(x)$ contains an invariant measure $\mu_t(x)$ such that

$$\text{dist}^*(\mu_t(x), \delta_{x_1}) = t \text{dist}^*(\delta_{x_2}, \delta_{x_1}). \quad (8)$$

After Poincaré recurrence Theorem μ_t is supported on $\{x_1, x_2\}$, so it is a convex combination of δ_{x_1} and δ_{x_2} . But the unique such convex combination satisfying

Equality (8) is $\mu_t = t\delta_{x_1} + (1+t)\delta_{x_2}$ (if the metric dist^* is chosen to depend linearly on t for the measures in the segment $[\delta_{x_1}, \delta_{x_2}]$). So $\mathcal{O}_{f|\overline{V}} = [\delta_{x_1}, \delta_{x_2}]$, as wanted. \square

Existence of the case (B) of Example 7.2

For all $0 < t < 1$ there exists an homeomorphism f as in Example 7.2, for which

$$\mathcal{O}_{f|\overline{V}} = \{t\delta_{x_1} + (1-t)\delta_{x_2}\}.$$

Proof. Applying similar arguments to those of the proof of case (A), let us construct f weakly dissipative near $W^s(x_2) \setminus (N_1 \cup N_2)$ and also weakly dissipative near $W^s(x_1) \setminus (N_1 \cup N_2)$. Precisely, let us denote $d_i(x)$ and $d'_i(x)$ the distances defined in Equalities (7) in the proof of case (A). We can C^0 perturb a map f (that is are preserving map in the neighborhoods U_1 and U_2), so that

$$\frac{d'_i}{3} \leq d_{i+1} \leq \frac{d'_i}{2}, \quad \frac{d_i}{3} \leq d'_i \leq \frac{d_i}{2} \quad \forall i \geq 1.$$

Adapting standard computations after applying Hartman-Grossman Theorem inside the neighborhoods U_1 and U_2 of the two saddles, we deduce that the staying times $N_i(1)$ and $N_i(2)$ (during the i -th visit to U_1 and U_2 respectively) satisfy the following inequalities, for some positive constants c and $k'(x)$:

$$N_i(1) \leq c \frac{\log d'_i}{\log \sigma_1} \leq k'(x) \frac{i}{\log \sigma_1} \leq N_i(1) + 1 \quad \forall i \geq 1,$$

$$N_i(2) \leq c \frac{\log d_i}{\log \sigma_1} \leq k'(x) \frac{i}{\log \sigma_2} \leq N_i(2) + 1 \quad \forall i \geq 1,$$

where $\sigma_{1,2} > 1$ are the expanding eigenvalues of the saddles $x_{1,2}$ respectively.

After similar computations to those in the proof of case (A), we deduce that the frequencies $\omega_n(U_1)$ and $\omega_n(U_2)$ of visits of the finite piece of orbit up to any stopping time $n \geq 1$, to the neighborhoods U_1 and U_2 respectively, can be computed as follows:

$$\omega_n(U_{1,2}) \sim \frac{\sum_{i=1}^m N_i(1, 2)}{km + \sum_{i=1}^m N_i(2) + \sum_{i=1}^m N_i(1)}$$

where k is a constant and m is the number of visits to U_2 up to time n . Thus,

$$\frac{1}{\omega_n(U_1)} \sim 1 + \frac{km \log \sigma_1}{k'(x) \sum_{i=1}^m i} + \frac{\log \sigma_1}{\log \sigma_2} \rightarrow 1 + \frac{\log \sigma_1}{\log \sigma_2}$$

and analogously

$$\frac{1}{\omega_n(U_2)} \rightarrow 1 + \frac{\log \sigma_2}{\log \sigma_1}.$$

After checking that $1 = (1 + \log \sigma_1 / \log \sigma_2)^{-1} + (1 + \log \sigma_2 / \log \sigma_1)^{-1}$ we deduce that the empirical sequence (1) will be convergent to

$$t\delta_1 + (1 - t)\delta_2, \text{ where } t = \frac{1 + \log \sigma_2 / \log \sigma_1}{2 + \log \sigma_2 / \log \sigma_1 + \log \sigma_1 / \log \sigma_2}.$$

Since the eigenvalues $\sigma_{1,2} > 1$ can be arbitrarily chosen, the parameter t can be equalled to any previously specified value in the open interval $(0, 1)$. \square

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